On the average nodal volume for different invariant random polynomials

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- Brief history (polynomials and Laplace–Beltrami eigenfunctions).
- Isotropy irreducible homogeneous spaces (expectations).
- The case of spheres (comparison of the Kostlan–Shub–Smale and $L^2(S^m)$ models).
- Isotropy irreducible homogeneous spaces (variances).

Brief history (polynomials)

- Paley, Wiener, Zygmund (1932): random functions.
- Bloch and Polya (1932):

$$u = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n = 0,$$

  $a_k = 0, \pm 1$ with probability $\frac{1}{3}$. Mean number of real roots is $O\left(\sqrt{n}\right)$.
- Littlewood, Offord (1938, 1939): other distributions and better estimates.

Brief history (polynomials)

- Mark Kac (1943): The first exact formula and asymptotically sharp estimate to the mean number of real roots for standard Gaussian coefficients a_j: it is asymptotic to $\frac{2}{\pi}\ln n$, has the upper bound $\ln n + \frac{14}{\pi}$, $n \geq 2$, and is subject to the formula

$$\mathrm{M}(N_u) = \frac{4}{\pi}\int_0^1 \frac{\sqrt{1-\Phi_n(x)^2}}{1-x^2}\,dx,$$

where $\Phi(x) = (1+n)x^n\frac{1-x^2}{1-x^{2n}}$.

- Ibragimov I.A., Maslova N.B.: other distributions, estimates of the variance.

Brief history (polynomials)

In early 90th, Kostlan realized the geometric meaning of the computation of $\mathrm{M}(N_u)$.

- Let $\gamma(t) = (1, t, t^2, \ldots, t^n)$ be the moment curve and $a = (a_0, \ldots, a_{n-1}, a_n)$ be a vector in $\mathbb{R}^{n+1}$. The points in $\gamma \cap a^\perp$ are in one-to-one correspondence with the zeroes of the polynomial with the coefficients a.
- The same is true for the curve $\tilde{\gamma}(t) = \frac{\gamma(t)}{|\gamma(t)|}$ in the unit sphere S^n.
- Integrating over S^n on a, we get the expectation of zeroes. Due to a Crofton type formula (next page), it is proportional to the length of $\tilde{\gamma}$.

The arguments above can be extended onto the multidimensional case and other distributions.

A kinematic formula

The following formula is a particular case of Theorem 3.2.48 in Federer's book on Geometric Measure Theory. Let $A, B \subseteq S^d$ be compact, A be k-rectifiable, and B be l-rectifiable ("k-rectifiable" means "Lipschitz image of a bounded subset of $\mathbb{R}^k$"). Set $r = k + l - d$. Suppose $r \geq 0$. Then

$$\int_{\mathrm{O}(d+1)} \mathfrak{h}^r(A \cap gB)\, dg = \frac{\varpi^r}{\varpi_k \varpi_l} \mathfrak{h}^k(A)\mathfrak{h}^l(B),$$

where $\varpi_k = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}$ is the volume of S^k.

The Kostlan–Shub–Smale model

Let $\mathcal{P}_{n,m}$ be the space of homogeneous polynomials of degree n on $\mathbb{R}^{m+1}$ and let the inner product in it be defined by the condition of orthogonality of monomials x^α and the equality

$$|x^\alpha|^2 = \alpha!,$$

where $x \in \mathbb{R}^{m+1}$, $\alpha \in \mathbb{Z}_+^n$. Let u_k be a random polynomial in $\mathcal{P}_n$ which is subject to the distribution with the density $\pi^{-\frac{d}{2}} e^{-|u|^2}$ in $\mathcal{P}_n$, and $d = \dim \mathcal{P}_n = \binom{n+m}{m}$. Kostlan found the expectation of the number of roots for the system $u_k = 0$, where $k = 1, \dots, m$ (the roots are counted in the projective space). It is equal to $n^{\frac{m}{2}}$. In 90th, Smale and Shub extended Kostlan's result onto the case of system $u_k = 0$ of different degrees $n_1, \dots, n_m$. The answer is $\sqrt{n_1 \dots n_m}$.

Expectation of the Euler characteristic

- Podkorytov S.S. (1999). Let u be a Gaussian random polynomial of degree n on $\mathbb{R}^{m+1}$. Set $r = \mathrm{M}\left(\frac{\partial u}{\partial x_m}(o)^2\right)/\mathrm{M}(u(o)^2)$, where $o = (1, 0, \ldots, 0)$,

$$I_m(t) = \int_0^t (1 - x^2)^{\frac{m-1}{2}}\, dx, \quad \mu_m(r) = \frac{I_m\left(\sqrt{r}\right)}{I_m(1)}.$$

Then

$$\mathrm{M}\left(\chi(N_u)\right) = \mu_n(r),$$

where $N_u = u^{-1}(0)$ and m is odd. Moreover,

$$\frac{1 - (-1)^m}{2} \le r \le \frac{n(n + m - 1)}{m}.$$

Further results

In 2007, Bürgisser extended Podkorytov's theorem onto the case of higher codimension. Let $n - k$ be even, $f = (f_1, \dots, f_k)$ be Gaussian random polynomials, $r = (r_1, \dots, r_k)$ be their Podkorytov's parameters. The expectation of $\chi(N_f)$ depends only on r and dimensions. He derived a formula for the expectation. His proof involves Weyl's Tube Formula.
Wschebor (2005) found an upper bound for the variance of the number of roots.

Random polynomials

Notation

In what follows,

- G is a compact Lie group,
- M is a connected homogeneous space of G,
- o is the base point of M and H is its stable subgroup.

We say that a function u on M is a polynomial if the linear span of its translates $u \circ g$, $g \in G$, is finite dimensional.

- $\mathcal{E}$ is a finite dimensional G-invariant subspace of continuous functions equipped with a G-invariant inner product $\langle \ , \ \rangle$,
- $\mathcal{S}$ is the unit sphere in $\mathcal{E}$,
- $m = \dim M$,
- $d = \dim \mathcal{S} = \dim \mathcal{E} - 1$.

Hausdorff measures

The *Hausdorff measure* $\mathfrak{h}^s$ of dimension s is defined in two steps:

1) Let $\delta > 0$ and

$$\mathfrak{h}^s_\delta(E) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s+1}{2}\right)} \inf\left\{\sum \left(\frac{\operatorname{diam} C}{2}\right)^s : E \subseteq \bigcup C, \quad \operatorname{diam} C < \delta\right\};$$

2) set $\mathfrak{h}^s(E) = \sup_{\delta>0} \mathfrak{h}^s_\delta(E)$.

The measure $\mathfrak{h}^0$ is the counting function (i.e., $\mathfrak{h}^0(E) = \operatorname{card}(E)$).

Isotropy irreducible homogeneous spaces

If H acts in T_oM irreducibly, then M is called isotropy irreducible.

- Let N be a Riemannian G-manifold and $\iota : M \to N$ be an equivariant nonconstant smooth map. Then ι is a local diffeomorphism and a finite covering.
- The invariant Riemannian metric in M is unique up to a scaling factor. Hence the restriction of the Riemannian metric in N onto $\iota(M)$ is proportional to that of M.

- Let s be the coefficient of proportionality. If γ is a path in M of length l, then the path $\iota \circ \gamma$ has length sl. It follows that ι is a local metric homothety and the same is true for the Hausdorff measure $\mathfrak{h}^k$, with the coefficient s^k.
- By definition,

$$s = \frac{|d_p\varphi(v)|_N}{|v|_M}$$

for any $p \in M$ and $v \in T_pM$.

The immersion $M \to \mathcal{S}$

There is a natural equivariant mapping $\iota : M \to \mathcal{S}$. Let $p \in M$ and $\phi_p \in \mathcal{E}$ be such that $u(p) = \langle \phi_p, u \rangle$ for all $u \in \mathcal{E}$. Set

$$\iota(p) = \frac{\phi_p}{|\phi_p|}.$$

Lemma

If $|\ |$ is the norm of $L^2(M)$, then $s = \frac{|d_o\iota(v)|_{\mathcal{E}}}{|v|_{T_oM}} = \frac{1}{c}\sqrt{\frac{-\operatorname{Tr}\Delta}{m}}$, where Δ is the Laplace–Beltrami operator on M. If $\mathcal{E}$ is an eigenspace of Δ, then

$$s = \sqrt{\frac{\lambda}{m}},$$

where λ is the eigenvalue of $-\Delta$ in $\mathcal{E}$.

Connection between coefficients for different norms

Let $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l$, where $\mathcal{E}_j$ are irreducible, $|\ |$ be the $L^2(M)$ norm and $\widetilde{|\ |}$ be another G-invariant norm. Then for all $u, v \in \mathcal{E}$ we have

$$\widetilde{\langle u, v \rangle} = \tau_1^{-1} \langle u_1, v_1 \rangle + \cdots + \tau_l^{-1} \langle u_l, v_l \rangle \,, \tag{1}$$

where u_j, v_j are components of u, v in the decomposition.

Proposition

We have

$$\begin{aligned} \tilde{c}^2 = \widetilde{|\tilde{\phi}_p|}^2 = \tau_1 c_1^2 + \cdots + \tau_l c_l^2, \\ c^2 = |\phi_p|^2 = c_1^2 + \cdots + c_l^2, \\ \tilde{s}^2 = \nu_1 s_1^2 + \cdots + \nu_l s_l^2, \end{aligned}$$

where $\nu_j = \frac{\tau_j c_j^2}{\tilde{c}^2}$ and $s_j^2 = \frac{\lambda_j}{m}$, $j = 1, \ldots, l$.

Computation of expectations

Lemma

Let $X \subseteq M$ be $(r+1)$-rectifiable, where $r \le m-1$. Then

$$\int_{\mathcal{S}} \mathfrak{h}^r(N_u \cap X)\, du = \frac{\varpi_r}{\varpi_{r+1}} s\, \mathfrak{h}^{r+1}(X),$$

where du stands for the probability invariant measure on $\mathcal{S}$.

Let every space $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be as $\mathcal{E}$ above, $\mathbf{u} = (u_1, \ldots, u_k)$, and $N_{\mathbf{u}} = N_{u_1} \cap \ldots \cap N_{u_k}$, $k \le m$. Averaging over $\mathcal{S}_1 \times \cdots \times \mathcal{S}_k$, we get

$$\mathrm{M}\left(\mathfrak{h}^{m-k}(N_{\mathbf{u}})\right) = \varpi \frac{\varpi_{m-k}}{\varpi_m} s_1 \ldots s_k,$$

where $\varpi = \mathrm{Vol}(M)$. For $k = m$ we get the mean number of solutions to the system $u_i(p) = 0$, $i = 1, \ldots, k$.

Coefficients for the norm of $L^2(M)$

There is the well known decomposition

$$\mathcal{P}_n = \sum_{\substack{0\le j\le n,\\ n-j \text{ even}}} |x|^{n-j}\mathcal{H}_j,$$

where H_j is the space of harmonic homogeneous polynomials of degree j restricted to S^m. For the norm of $L^2(M)$ we have

$$c^2 = \dim \mathcal{P}_n = \sum_{\substack{0\le j\le n,\\ n-j \text{ even}}} c_j^2 = \binom{m+n}{m}$$

$$c_j^2 = \dim \mathcal{H}_j = \frac{(m+j-2)!(m+2j-1)}{(m-1)!j!},$$

$$s_j^2 = \frac{j(m+j-1)}{m},$$

$$s^2 = \frac{1}{c^2} \sum_{\substack{0\le j\le n,\\ n-j \text{ even}}} c_j^2 s_j^2 = \frac{n(m+n+1)}{m+2}.$$

Coefficients for the Kostlan–Shub–Smale model

The coefficients are subject to the formulas

$$\tau_j^{-1} = \frac{2^n}{\Gamma\left(\frac{m-1}{2}\right)} \Gamma\left(\frac{n-j+2}{2}\right) \Gamma\left(\frac{m+n+j+1}{2}\right),$$

$\tilde{c}^2 = \frac{1}{n!}$, $\tilde{s}^2 = n$. Thus $\nu_j = n! \tau_j c_j^2$. Set

$$\mu_n = \sqrt{(m-1)n}.$$

Theorem

The coefficients ν_j extends onto $\mathbb{C}$ as an entire function. On the interval $(0, n)$ the function $\ln \nu(x)$ is strictly concave and has the unique critical point x_c which corresponds to the global maximum on $(0, n)$. Moreover, if m is fixed and n is sufficiently large, then

$$\mu_n - \frac{m+1}{2} < x_c < \mu_n + 2.$$

Asymptotic behavior of ν as $n \to \infty$

In what follows, we assume m fixed. Also, ν is extended from $(0, n)$ onto $\mathbb{R}$ by zero. Set $\bar{\nu}_n = \nu(x_c) = \max\{\nu(t) : t \in \mathbb{R}\}$.

Theorem

For any $t > 0$

$$\lim_{n\to\infty} \frac{\nu(\mu_n t)}{\bar{\nu}_n} = \left(t^2 e^{1-t^2}\right)^{\frac{m-1}{2}}$$

where the sequence on the left converges uniformly on $(0, \infty)$. Moreover,

$$\bar{\nu}_n = \frac{A_m}{\sqrt{n}}(1 + o(1)),$$

where $A_m = \frac{2\sqrt{2}}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{m-1}{2e}\right)^{\frac{m-1}{2}}$.

The rate of decreasing

The function $\left(t^2 e^{1-t^2}\right)^{\frac{m-1}{2}}$ gives an upper bound for ν:

$$\frac{\nu(t\mu_n + 2)}{\nu(\mu_n)} < \left(t^2 e^{1-t^2}\right)^{\frac{m-1}{2}} .$$

Thus the coefficients $\nu_j = n!\,\tilde{c}_j^2$ decrease very fast when j grows; however, for large $j \leq n$ the estimate above is not sharp and the rate of decay is greater. A short table below illustrates this. Let $m = 10$ and $n = 900$. Then $\mu_n = 90$, the maximum of ν_j (over all j between 0 and n such that $n - j$ is even) is approximately 0.038 and is attained at $j = 86$. In the last row, b_j is the bound for ν_j defined by the inequality above (we multiply it on ν_{86}, replace t with $\frac{j}{\mu_n}$, and shift the index by 2).

j	100	200	450	700	900
ν_j	0.032	$6.0 \cdot 10^{-7}$	$2.9 \cdot 10^{-45}$	$9.0 \cdot 10^{-127}$	$6.1 \cdot 10^{-259}$
b_j	0.036	$1.4 \cdot 10^{-6}$	$2.4 \cdot 10^{-42}$	$9.8 \cdot 10^{-110}$	$9.1 \cdot 10^{-186}$

Approximation by polynomials of degree less than n

For $u \in L^2(S^m)$, let $\delta(u, V)$ be the distance in $L^2(S^m)$ from u to V.

Theorem

Let u be a random polynomial uniformly distributed in the unit sphere $\widetilde{\mathcal{S}} \subseteq \mathcal{P}_n$ for the norm $\widetilde{|\,|}$. If $\kappa > 0$, then for all sufficiently large n and $l > (\kappa + 1)\mu_n$

$$\mathrm{M}(\delta(u, \mathcal{P}_l)^2) < \frac{5}{2\sqrt{m-1}} \frac{e^{-\kappa^2}}{\kappa} \mathrm{M}(|u|^2).$$