On the classification of partially commutative groups up to universal equivalence

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Tarski and the beginning of model theory

Alfred Taski (1901-1983): Model-theoretic study of algebraic structures.

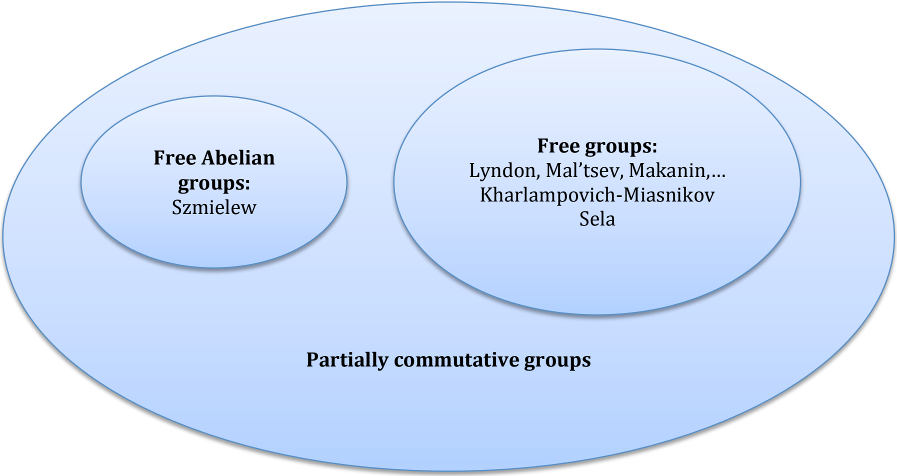
Model theory of groups

Model theory of groups

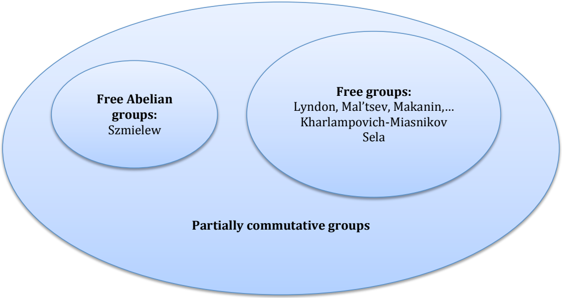
Abelian groups:
Szmielew

Free groups:
Lyndon, Mal'tsev, Makanin,...
Kharlampovich-Miasnikov
Sela

Model theory of groups



Model theoretic study of partially commutative groups



Goal

Model-theoretic study of the class of partially commutative groups (pc groups): a class that interpolates between free and free abelian groups.

Partially commutative groups

Definition

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a (undirected) simplicial graph. The partially commutative group (pc group) $\mathbb{G} = \mathbb{G}(\Gamma)$ defined by the commutation graph Γ is the group given by the following presentation,

$$\mathbb{G} = \langle V(\Gamma) \mid [v_1, v_2] = 1, \text{ whenever } (v_1, v_2) \in E(\Gamma) \rangle.$$

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Examples

$\mathbb{G}($  $) = F_3$

$\mathbb{G}($  $) = \mathbb{Z}^3$

$\mathbb{G}($  $) = F_2 \times F_2$

$\mathbb{G}($  $) = \mathbb{Z}^2 * \mathbb{Z}^2$

$\mathbb{G}($  $) = \langle x_0, \ldots, x_4 \mid [x_i, x_{i+1}] = 1, \ i \in \mathbb{Z}_5 \rangle$

Partially commutative groups

Computer Science

- Concurrent systems, Petri nets (theory of traces)
- Rearrangements of strings

Algebra

- Magnus transformation
- Partially commutative structures
- Homological finiteness properties
- Algorithmic problems

Partially Commutative Groups

$$\langle x_1, \ldots, x_n \mid x_i x_j = x_j x_i \text{ for some pairs } (i,j) \rangle$$

Group Theory and Topology

- Theory of CAT(0) and Special Cube Complexes
- Virtually Haken Conjecture
- Virtually Fibred Conjecture

Physics

- Modelling growth of soot
- Statistical physics (statistical invariants of braids)

Universal equivalence

Main Question

When is $\mathbb{G}(\Delta) \equiv_\forall \mathbb{G}(\Gamma)$?

Group theoretic re-formulation

Characterisation (Remeslennikov '89, Daniyarova-Miasnikov-Remeslennikov '08)

$\mathbb{G}(\Delta) \equiv_\forall \mathbb{G}(\Gamma)$ if and only if $\mathbb{G}(\Delta)$ and $\mathbb{G}(\Gamma)$ discriminate each other.

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Definition

A group G is discriminated by H if for any finite set $S \subset G$ there is a homomorphism $\varphi_S : G \to H$ that it is injective on S.

Learning from abelian groups

Example

$\mathbb{G}(\Delta) \equiv_\forall \mathbb{Z}$ if and only if $\mathbb{G}(\Delta) \simeq \mathbb{Z}^m$, $m \geq 1$.

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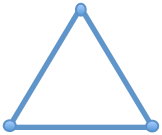
Idea

We need to "increase abelians".

Learning from abelian groups

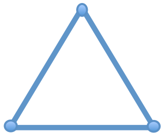
$\mathbb{Z}$



$\mathbb{Z}^3$

Learning from abelian groups

$\mathbb{Z}$



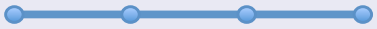
$\mathbb{Z}^3$

F_2

$\mathbb{Z}^2 * \mathbb{Z}^2$

Definition

Graph theory: from a graph to its inflation.



$G(P_4)$

$G(P_4^2)$

Increasing abelians

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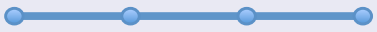

Definition

We denote by $\mathbb{G}(\Delta^n)$ the graph product $\mathcal{G}(\Delta, \mathbb{Z}^n)$. Note that $\mathbb{G}(\Delta^n)$ is a pc group.

Increasing abelians

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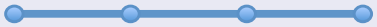

Proposition (C., 2015)

$\mathbb{G}(\Delta^n) \equiv_\forall \mathbb{G}(\Delta)$.

Increasing abelians

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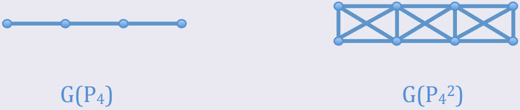

Proposition (C., 2015)

$\mathbb{G}(\Delta^n) \equiv_\forall \mathbb{G}(\Delta)$.

Characterisation?

Is the proposition a characterisation: $\mathbb{G}(\Gamma) \equiv_\forall \mathbb{G}(\Delta)$ if and only if $\mathbb{G}(\Gamma) = \mathbb{G}(\Delta^n)$ for some $n \in \mathbb{N}$? No...

Learning from free groups: Mutual Embeddability

Example

$F_2 < F_m < F_2$ and so $\mathrm{Th}_{\forall}(F_2) = \mathrm{Th}_{\forall}(F_m)$, $m \geq 2$.

Example

$\mathbb{G}(P_4) < \mathbb{G}(P_5) < \mathbb{G}(P_4)$ and so $\mathrm{Th}_{\forall}(\mathbb{G}(P_4)) = \mathrm{Th}_{\forall}(\mathbb{G}(P_5))$.

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Main result

So we have to "operations"

- increase abelians
- mutual embeddability

and both are necessary. Are they sufficient?

Main result

Theorem (C., 2015)

$$\mathbb{G}(\Delta) \equiv_{\forall} \mathbb{G}(\Gamma)$$

if and only if

$$\mathbb{G}(\Delta) \hookrightarrow \mathbb{G}(\Gamma^n)$$

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where $n = |V(\Delta)||V(\Gamma)|$.

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The model theoretic problem of classification upto universal equivalence reduces to embeddability of pc groups.

Embeddability Problem between pc groups

Question

Can we characterise when $\mathbb{G}(\Delta) < \mathbb{G}(\Gamma)$?

Embeddability Problem between pc groups

Dream

Find a condition which determines when $\mathbb{G}(\Delta) < \mathbb{G}(\Gamma)$ and express this condition as a universal sentence.

Strategy

Proposition, C., 2015

Given Δ and Γ, there exists a finite set of commutators $\mathcal{C}(\Delta, \Gamma)$ that "witnesses" the existence of an embedding from $\mathbb{G}(\Delta)$ to $\mathbb{G}(\Gamma)$.

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More precisely, there exists a finite set of commutators $\mathcal{C}$ such that

- a map $\varphi : \mathbb{G}(\Delta) \to \mathbb{G}(\Gamma)$ is an embedding if and only if
- φ is injective in the set $\mathcal{C}$.

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Counterexample

Any map $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ is injective in the set of commutators but it is NOT an embedding!

Strategy

Require no repetition of letters in the image on generators. This can be assumed if one allows to increase abelians.

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$$\begin{array}{llll} \varphi : & \mathbb{Z}^2 =< x, y > & \to & \mathbb{Z} =< a > \\ & x & \to & a^n \\ & y & \to & a^m \end{array}$$

Note that $\alpha(\varphi(x)) = \{a\} = \alpha(\varphi(y))$. Consider

$$\begin{array}{llll} \varphi' : & \mathbb{Z}^2 =< x, y > & \to & \mathbb{Z}^2 =< a, A > \\ & x & \to & a^n \\ & y & \to & A^m \end{array}$$

Strategy

Definition (Tame Embedding)

An embedding $\varphi : \mathbb{G}(\Delta) \to \mathbb{G}(\Gamma)$ induced by the map $x \to w_x(Y)$ is tame if $\alpha(\varphi(x)) \cap \alpha(\varphi(x')) = \emptyset$.

Strategy

Proposition, C., 2015

More precisely, there exists a finite set of commutators $\mathcal{C}$ such that

- a TAME map $\varphi : \mathbb{G}(\Delta) \to \mathbb{G}(\Gamma)$ is an embedding if and only if
- φ is injective in the set $\mathcal{C}$.

Strategy

Proposition, C., 2015

Given Δ and Γ, there exists a finite set of commutators $\mathcal{C}(\Delta, \Gamma)$ that "witnesses" the existence of a tame embedding from $\mathbb{G}(\Delta)$ to $\mathbb{G}(\Gamma)$.

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Remark

The set $\mathcal{C}(\Delta, \Gamma)$ is not as easy as one may expect: for any weight n, there exist Δ, Γ such that $\mathcal{C}$ contains commutators of weight more than n.

Corollary: Algorithmic approach

Embeddability

Graph embedding induces group embedding:

Conjugation "trick":

Corollary: Algorithmic approach

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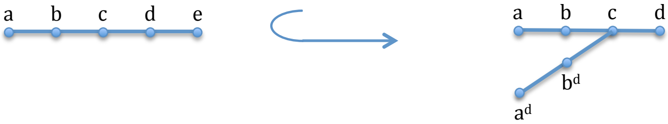

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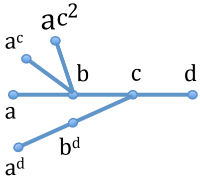
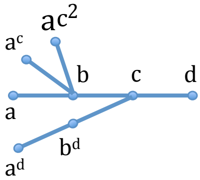


Definition

Let Γ be a simplicial graph, $V(\Gamma) = \{a_1, \dots, a_k\}$. The extension graph Γ^e is defined as follows:

1. $V(\Gamma^e) = \{a_i^w \in \mathbb{G}(\Gamma) \mid w \in \mathbb{G}(\Gamma)\}$ and
2. $E(\Gamma^e) = \{(u, v) \in V(\Gamma^e) \times V(\Gamma^e) \mid [u, v] = 1 \text{ in } \mathbb{G}(\Gamma)$.

Corollary: Algorithmic approach



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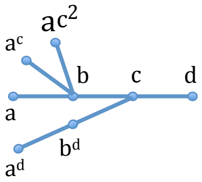
Remark

Note that in general the extension graph is an infinite (not locally finite) graph.

Corollary: Algorithmic approach



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Extension Graph Conjecture (Kim-Koberda)

$\mathbb{G}(\Delta) < \mathbb{G}(\Gamma)$ if and only if $\Delta < \Gamma^e$.

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Theorem, C., 2015

Given Δ and Γ one can effectively construct a finite graph $\Gamma_R < \Gamma^e$ such that:

- $\Delta < \Gamma^e$ if and only if
- $\Delta < \Gamma_R$

Corollary: Algorithmic approach

Extension Graph Conjecture (Kim-Koberda)

$\mathbb{G}(\Delta) < \mathbb{G}(\Gamma)$ if and only if $\Delta < \Gamma^e$.

Corollary, C., 2015

It is decidable whether or not $\Delta < \Gamma^e$ and can be expressed with a universal sentence.

Corollary: Algorithmic approach

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Theorem (Kim-Koberda, 2011)

If $\Delta < \Gamma^e$ then $\mathbb{G}(\Delta) < \mathbb{G}(\Gamma)$.

Corollary: Algorithmic approach

Extension Graph Conjecture (Kim-Koberda)

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Theorem (C.-Duncan-Kazachkov, 2013)

The Extension Graph Conjecture is false: There exist Δ and Γ such that $\mathbb{G}(\Delta) < \mathbb{G}(\Gamma)$ and $\Delta \not< \Gamma^e$.

Theorem (Kim-Koberda, 2011)

If Γ is triangle-free, then the Extension Graph Conjecture holds.

Theorem (C.,Duncan, Kazachkov, 2013)

If Γ is triangle-built, then the Extension Graph Conjecture holds.

Corollary: Algorithmic approach

Corollary(C., 2015)

The Embeddability Problem is decidable for 2-dimensional pc groups: there exists an algorithm that given an arbitrary graph Δ and a triangle-free graph Γ decides whether or not $\mathbb{G}(\Delta) < \mathbb{G}(\Gamma)$.

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There is an algorithm that given Δ and a triangle-free graph Γ determines whether or not $\mathbb{G}(\Delta) \equiv_{\forall} \mathbb{G}(\Gamma)$.

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Corollary: Model-theoretic approach

Corollary(C.)

Let C_n denote a cycle (atomic graph) $n \geq 3$. Then the following are equivalent:

- $\mathbb{G}(C_n)$ and $\mathbb{G}(C_m)$ are elementary equivalent;
- $\mathbb{G}(C_n)$ and $\mathbb{G}(C_m)$ are universally equivalent;
- $\mathbb{G}(C_n)$ and $\mathbb{G}(C_m)$ are isomorphic;
- $n = m$.

Open questions

1. Algebraic: When $\mathbb{G}(\Delta) < \mathbb{G}(\Gamma)$?
2. We can formulate the above questions in the context of Lie Algebras: Given pc Lie algebras $\mathcal{L}(\Delta), \mathcal{L}(\Gamma)$, when $\mathcal{L}(\Delta) < \mathcal{L}(\Gamma)$
3. Algorithmic: Is the tame embeddability problem decidable? Is the embeddability problem decidable?
4. Model-theoretic: Description of pc groups elementarily equivalent to a cycle (or atomic graph): $\mathbb{G}(\Delta) \equiv \mathbb{G}(C_n)$ if and only if $\mathbb{G}(\Delta)$ is a hairy n cycle;

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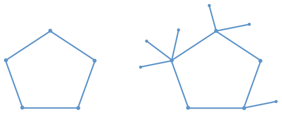

СПАСИБО ЗА ВНИМАНИЕ!

THANK YOU!