On invariants of probability spaces

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By a probability logic we shall mean a triple $\langle \mathscr{L}, \mathcal{K}, \Vdash \rangle$ where:

$\mathscr{L}$ is a formal language, intended for building $\mathscr{L}$-formulas;

$\mathcal{K}$ is a class of $\mathscr{L}$-structures, described using probability spaces;

$\Vdash$ is a satisfiability relation between $\mathscr{L}$-structures and $\mathscr{L}$-formulas.

Furthermore we shall limit ourselves to logics with *quantifiers*.

I shall propose the probability logic QPL — which can be viewed as extending a variant of the well-known quantifier-free ‘polynomial’ logic from [Fagin et al. 1990] by adding ‘quantifiers over events’.

In the framework of QPL we can (among other things)

- safely pass from a probability space to its quotient modulo events of measure zero, and vice versa;
- define the properties of being finite, discrete and atomless, modulo events of measure zero, for probability spaces;
- describe a useful classification of probability spaces reminiscent of the famous elementary classification of Boolean algebras;
- obtain some interesting results concerning the connection between computability and continuity in a probabilistic setting.

Elements of $\mathscr{X} := \{x_i \mid i \in \mathbb{N}\}$ are called variables.

By the set of e-terms we mean the smallest set containing $\mathscr{X}$, s.t. if t_1 and t_2 are e-terms, then $\overline{t_1}$, $t_1 \cap t_2$ and $t_1 \cup t_2$ are e-terms.

Definition

An atomic QPL-formula is an expression of the form

$$f\left(\mu\left(t_1\right), \ldots, \mu\left(t_n\right)\right) \leqslant g\left(\mu\left(t_{n+1}\right), \ldots, \mu\left(t_{n+k}\right)\right)$$

where f and g are polynomials with coefficients in $\mathbb{Q}$, μ is a fixed symbol and $t_1, \ldots, t_{n+k}$ are e-terms.

By an e-quantifier we mean Qx with Q $\in \{\forall, \exists\}$ and $x \in \mathscr{X}$.

Let $\wedge$, $\vee$, $\neg$ and $\rightarrow$ denote the usual logical connectives.

Now QPL-formulas are built up from atomic QPL-formulas using connective symbols and e-quantifiers in the standard way.

Definition

By a QPL-structure we simply understand a pair $(\mathscr{P}, v)$ where:

- $\mathscr{P}$ is a probability space, i.e. it has the form $\langle \Omega, \mathscr{A}, \mathrm{P} \rangle$ where
  - — $\mathscr{A}$ is a σ-algebra over a non-empty set Ω, and
  - — P is a countably additive probability measure on $\mathscr{A}$.
- v is a partial valuation, i.e. a mapping from a subset of $\mathscr{X}$ to $\mathscr{A}$.

Next we easily describe the satisfiability relation for our logic.

Inductively assign elements of $\mathscr{A}$ to all e-terms over $\mathrm{dom}\,(v)$:

$$\begin{aligned} v\,(\overline{t_1}) &:= \text{ the complement of } v\,(t_1) \text{ in } \Omega; \\ v\,(t_1 \cap t_2) &:= \text{ the intersection of } v\,(t_1) \text{ and } v\,(t_2); \\ v\,(t_1 \cup t_2) &:= \text{ the union of } v\,(t_1) \text{ and } v\,(t_2). \end{aligned}$$

For every quantifier-free QPL-formula Ψ with $FV\,(\Psi) \subseteq \mathrm{dom}\,(v)$,

$$(\mathscr{P}, v) \Vdash \Psi \iff \text{the result of replacing each } \mu\,(t) \text{ in } \Psi \text{ by } \mathrm{P}\,(v\,(t)) \text{ holds in } \mathfrak{R} = \langle \mathbb{R}, +, \times, \leqslant \rangle.$$

Hence if we forget about e-quantifiers, then we get a variant of one decidable probability logic studied in [Fagin et al. 1990].

And I extend the above relation to all QPL-formulas whose free variables belong to $\mathrm{dom}(v)$ by requiring that:

- the logical connectives $\wedge$, $\vee$, $\neg$ and $\rightarrow$ behave clasically;
- variables in *e*-quantifiers range over 'events', i.e. elements of $\mathscr{A}$.

We need some notation. Given a probability space $\mathscr{P}$, let

$$\mathrm{Th}(\mathscr{P}) \;:=\; \{\Phi \mid \Phi \text{ is a QPL-sentence and } \mathscr{P} \Vdash \Phi\}.$$

Remark

In fact the results below remain true if we extend QPL by adding quantifiers over reals (and appropriately modifying the notion of a formula).

Consider a probability space $\mathscr{P} = \langle \Omega, \mathscr{A}, \mathrm{P} \rangle$. Denote

$$\Lambda \ := \ \{E \in \mathscr{A} \mid \mathrm{P}(E) = 0\}.$$

Clearly the subset relation and the equality relation on $\mathscr{A}$, modulo Λ, are defined in $\mathscr{P}$ by the following QPL-formulas:

$$x \preccurlyeq y \ := \ \mu(\overline{x} \cup y) = 1;$$
$$x \sim y \ := \ x \preccurlyeq y \wedge y \preccurlyeq x.$$

Obviously $x \sim y$ gives us a congruence relation for $(\mathscr{A}, \preccurlyeq)$. For each $E \in \mathscr{A}$ we let $[\![E]\!] := \{E' \in \mathscr{A} \mid \mathscr{P} \Vdash E \sim E'\}$.

Define the Boolean operations on $\mathscr{A}_\sim := \{[\![E]\!] \mid E \in \mathscr{A}\}$ by

$$\begin{aligned}\neg[\![E_1]\!] &:= [\![\Omega \setminus E_1]\!],\\ [\![E_1]\!] \wedge [\![E_2]\!] &:= [\![E_1 \cap E_2]\!],\\ [\![E_1]\!] \vee [\![E_2]\!] &:= [\![E_1 \cup E_2]\!].\end{aligned}$$

Now $\mathfrak{A} := \langle \mathscr{A}_\sim, \neg, \wedge, \vee \rangle$ turns out to be an 'abstract' σ-algebra, and the resulting mapping $\mathrm{P}_\sim : [\![E]\!] \mapsto \mathrm{P}\,(E)$ can hence be viewed as a countably additive measure on $\mathfrak{A}$; we leave $\Omega_\sim$ unspecified.

Remark

Of course, it is easy to adapt the initial QPL-semantics to deal with such 'abstract' probability spaces — like the one just described.

Let $\mathscr{P}_\sim$ denote $\langle \mathfrak{A}, \mathrm{P}_\sim \rangle$. As one easily checks, for every QPL-formula $\Phi(x_1, \ldots, x_n)$ and $\{E_1, \ldots, E_n\} \subseteq \mathscr{A}$ we have

$$\mathscr{P} \Vdash \Phi(E_1, \ldots, E_n) \quad \Longleftrightarrow \quad \mathscr{P}_\sim \Vdash \Phi([\![E_1]\!], \ldots, [\![E_n]\!]).$$

As for some primary classes of spaces, the Σ_2-QPL-sentence

$$\mathit{Fin} \;:=\; \exists x \forall y \left(\mu(x) > 0 \wedge (\mu(y) > 0 \rightarrow \mu(x) \leqslant \mu(y))\right)$$

defines the property of being finite — more precisely, it asserts the existence of the minimal non-zero value of our measure.

$$\mathscr{A}_\sim \text{ is finite} \quad \Longleftrightarrow \quad \mathit{Fin} \text{ holds in } \mathscr{P}.$$

The predicate '$[\![x]\!]$ is an atom of $\mathfrak{A}$' is defined in $\mathscr{P}$ by

$$At\,(x) \;:=\; \mu\,(x) > 0 \wedge \forall y\,((y \preccurlyeq x \wedge \mu\,(y) > 0) \to y \sim x).$$

Let $\mathscr{D}$ denote $\{[\![E]\!] \mid E \in \mathscr{A} \text{ and } \mathscr{P} \Vdash At\,(E)\}$. Note that $\mathscr{D}$ is at most countable; thus the supremum of $\mathscr{D}$ in $\mathfrak{A}$ exists.

Finally we can express the property of being discrete as

$$Dis \;:=\; \forall x\, \exists y\, (\mu\,(x) > 0 \to (y \preccurlyeq x \wedge At\,(y))).$$

Hence in QPL each $\mathscr{P}$ satisfying *Dis* can be identified with a discrete probability space (to which $\mathscr{P}_{\sim}$ is isomorphic).

Definition

The invariant of $\mathscr{P}$ is the function $\sharp_{\mathscr{P}}$ from $\mathbb{R}^+$ into $\mathbb{N}$ given by

$$\sharp_{\mathscr{P}}(r) \ := \ \text{the cardinality of } \{[\![E]\!] \in \mathscr{D} \mid \mathsf{P}(E) = r\}$$

where $[\![\cdot]\!]$ and $\mathscr{D}$ are as above. Note that $\sharp_{\mathscr{P}}(r) \leqslant \lfloor r^{-1} \rfloor$ for all $r \in \mathbb{R}^+$.

Theorem

For any two probability spaces $\mathscr{P}_1$ and $\mathscr{P}_2$ we have

$$\sharp_{\mathscr{P}_1} = \sharp_{\mathscr{P}_2} \quad \Longleftrightarrow \quad \mathrm{Th}(\mathscr{P}_1) = \mathrm{Th}(\mathscr{P}_2).$$

Obviously if $\mathscr{P}$ is atomless, then $\sharp_{\mathscr{P}}$ is simply the zero function.

In particular all atomless spaces have the same QPL-theory, that of the Lebesgue measure $\mathscr{L}$ on the unit interval. Moreover:

Theorem

$\mathrm{Th}(\mathscr{L})$ *is computable (i.e. algorithmically decidable).*

$\sharp_{\mathscr{P}}$ can be viewed as a 'structural representation' of $\mathscr{P}$, but not of its QPL-theory. E.g., given an integer $n \geqslant 2$, take

$$f_n(r) := \begin{cases} 1 & \text{if } r = n^{-k} \text{ for some } k \in \mathbb{N}^+ \\ 0 & \text{otherwise} \end{cases}$$

Proposition

Let $\mathscr{P}$ be a space with $\sharp_{\mathscr{P}} = f_n$. Then $\mathrm{Th}(\mathscr{P})$ is Π^1_∞-complete.

So even when we deal with quite simple representations, the theories of their spaces may turn out to be very complex.

Theorem

The validity problem for QPL *is* Π^1_∞*-complete.*

There are a partial function γ from $\mathcal{P}(\mathbb{N})$ onto the collection of all invariants and an effective τ translating each sentence Φ of QPL into a formula $\Phi^\tau(X)$ of monadic s.-o. arithmetic, s.t. for every $A \subseteq \mathbb{N}$,

$\mathfrak{N} \models \Phi^\tau(A) \iff$ there exists a probability space $\mathscr{P}$ for which $\sharp_{\mathscr{P}} = \gamma(A)$ and $\mathscr{P} \Vdash \Phi$.

Some references

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