Algebraic geometry over free left regular band

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Informal definition

Let A^* be the free semigroup of an alphabet $A = \{\text{all Russian letters}\}$. Any Russian word is an element of A^*. Let $A_1^* \subseteq A^*$ be the set of words with no repetitions.

Is the set A_1^* interesting?

Many popular (see TV) words do not contain letter repetitions: Путин, водка, крымнаш, Сталин, нефть, бандеровцы, Кадыров, рубль, демократия, Гейропа, духовныэ скрепы, ватник.

Amazingly, there exist semigroups that describe words with no letter repetitions.

A semigroup is called a left regular band (LRB) if the identities $xx = x$, $xyx = xy$ hold.

Definition

The free LRB $\mathcal{F}_n$ of rank n is the set of all words of the alphabet $\{a_1, a_2, \dots, a_n\}$ such that each word consists of different letters. E.g.

$$\mathcal{F}_3 = \{a_1, a_2, a_3, a_1a_2, a_2a_1, a_1a_3, a_3a_1, a_3a_2, a_2a_3,$$

$$a_1a_2a_3, a_1a_3a_2, a_2a_1a_3, a_2a_3a_1, a_3a_1a_2, a_3a_2a_1\}$$

The product of $w_1, w_2 \in \mathcal{F}_n$ is defined as follows:

$$w_1 \circ w_2 = w_1(w_2)^{\exists},$$

where the operator $\exists$ deletes all letters of w_2 which occur earlier. For example,

$$(a_1a_2)(a_2a_3a_1) = a_1a_2a_3.$$

Obviously, elements containing all letters are left zeroes:

$$(a_1a_2a_3)x = a_1a_2a_3 \text{ for all } x.$$

Partial orders over LRB-s

$$x \le y \Leftrightarrow xy = y$$

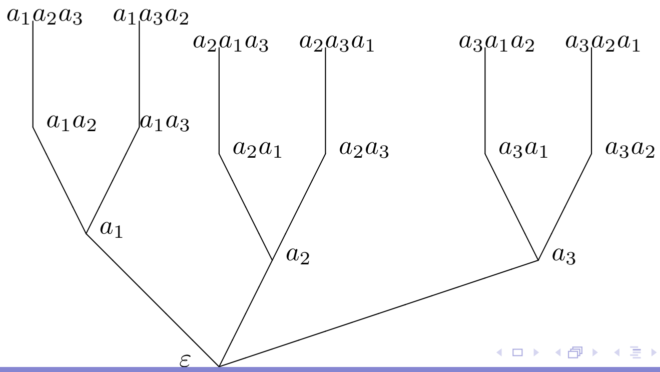
For elements of $\mathcal{F}_n$ $x \le y$ means that x is a prefix of y. E.g.

$$a_1a_2 \le a_1a_2a_3, a_1 \le a_1.$$

$\le$-comparable elements are always commute.

$\leq$-order over $\mathcal{F}_n$ is a tree

Let us adjoint an identity ε to $\mathcal{F}_n$, hence the Hasse diagram of $\leq$ is



Random walks

Tsetlin library: a shelf of books. Any book has a probability p_i. With the given probabilities we choose a book and put it at the front of the shelf. In is a random walk on the Cayley graph of free LRB $\mathcal{F}_n$, where n is the number of books. The transition of the walk is left multiplications.
See
K. S. Brown, Semigroups, rings, and Markov chains, J. Theoret. Probab., (2000), 13(3), 871–938.
for another applications of LRB in random walks.

Motivation: matroids

A matroid M is a pair $(E, \mathcal{I})$, where E is a set (called the ground set) and $\mathcal{I}$ is a family of subsets of E (called the independent sets) with the following properties:

1. Every subset of an independent set is independent, i.e., for each $A' \subset A \subset E$, if $A \in \mathcal{I}$ then $A' \in \mathcal{I}$ (hereditary property).
2. If A and B are two independent sets of $\mathcal{I}$ and $|A| > |B|$, then there exists an element in A that when added to B gives a larger independent set than B (exchange property).

Take three linearly independent vectors $E = \{v_1, v_2, v_3\}$ of some vector space. Then independent sets of the matroid $M = (E, \mathcal{I})$ are the following

$$\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}$$

The free matroid of rank n is the simplest matroid ever. It is isomorphic to the class of all independent sets generated by n vectors.

Problem

Are there other simple matroids?

Idea!

One can consider matroids as LRB-s to understand the complexity of matroids.

Let $M = (E, \mathcal{I})$ be a matroid and $\overrightarrow{\mathcal{I}}$ be the class of all ordered sets of $\mathcal{I}$. Define the multiplication over $\overrightarrow{\mathcal{I}}$ by

$$(v_1, v_2, \ldots, v_n)(u_1, u_2, \ldots, u_m) = (v_1, v_2, \ldots, v_n, u_{i_1}, u_{i_2}, \ldots, u_{i_k}),$$

where each u_{i_j} does not depend on the previous elements.
Thus, the class of ordered independent sets $\overrightarrow{\mathcal{I}}$ becomes an LRB. For example, $\overrightarrow{\mathcal{I}}$ of the matroid generated by linearly independent vectors v_1, v_2, v_3 is isomorphic to $\mathcal{F}_3$.

Let $\overrightarrow{\mathcal{I}}_n$ be all ordered independent sets of the free matroid of rank n. $\overrightarrow{\mathcal{I}}_n$ is isomorphic to $\mathcal{F}_n$.
Denote the free left regular band of countable rank by $\mathcal{F}$ (it contains all $\mathcal{F}_n$). Let us give the definition: "a finite matroid $M = (E, \mathcal{I})$ is simple if $\overrightarrow{\mathcal{I}}$ is embedded in $\mathcal{F}$".
Thus, we are going to study finite subbands of $\mathcal{F}$.

Properties of subbands in $\mathcal{F}$

Hasse diagram of the order $\leq$ is a tree for any subband S of $\mathcal{F}$. Bands with such property of $\leq$ are called right hereditary.

Are other properties?

Let $\alpha(x)$ be the ancestor of $x \in S$ relative the order $\leq$. Since the Hasse diagram of $\leq$ is a tree, $\alpha(x)$ is unique.

$$S_c = \{s \mid \alpha(s)c = sc, \alpha(s)\alpha(c) \neq s\alpha(c)\}.$$

We say that S has local linear order if each set S_c admits a linear order $\sqsubset_c$ such that

1. if $S_b \subseteq S_c$ and $x \sqsubset_b y$ then $x \sqsubset_c y$;
2. very complicated condition;
3. boring condition.

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Theorem

For a finite LRB S the following conditions are equivalent:

1. S is embedded into $\mathcal{F}$;
2. S is right hereditary and has a local linear order.
3. S is a coordinate band of an irreducible algebraic set over $\mathcal{F}$ (equations with no constants);

Idea of the proof

Let S be an LRB. Consider a congruence θ:

$$x \sim y \Leftrightarrow xy = x, yx = y.$$

The quotient map $\sigma\colon S \to S/\theta$ produces a semilattice $\sigma(S)$ (i.e. $\sigma(S)$ is commutative and idempotent). In the sequel we denote semilattice elements by bold letters.

Let $\mathbf{F}$ be a free semilattice of countable rank. Obviously, $\sigma(\mathcal{F}) = \mathbf{F}$. Free generators of $\mathcal{F}$ ($\mathbf{F}$) are denoted by $a_1, a_2, \ldots$ ($\mathbf{a}_1, \mathbf{a}_2, \ldots$) and we have $\sigma(a_i) = \mathbf{a}_i$.

How to define an embedding of S into $\mathcal{F}$?

Below we define an embedding $\lambda\colon S \to \mathcal{F}$.

1. Take $s \in S$;
2. get $\sigma(s) \in \sigma(S)$;
3. embed $\sigma(S)$ into $\mathbf{F}$;
4. now $\sigma(s) = \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$;
5. using local linear order, sort letters $\mathbf{a}_i$; for example, $\sigma(s) = \mathbf{a}_2\mathbf{a}_3\mathbf{a}_1$;
6. replace elements $\mathbf{a}_i \in \mathbf{F}$ to the corresponding generators $a_i \in \mathcal{F}$ and obtain
7. $\lambda(s) = a_2a_3a_1 \in \mathcal{F}$.