Searching for a Graph Isomorphism

G.A.Noskov

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Outline

- Formulation of the GI
- Searching for a GI
- SGI versus GI
- Algorithm for SGI
- Disease
- A gift
- Solving GI a.a.surely
- E-test
- Questions
- Conclusion

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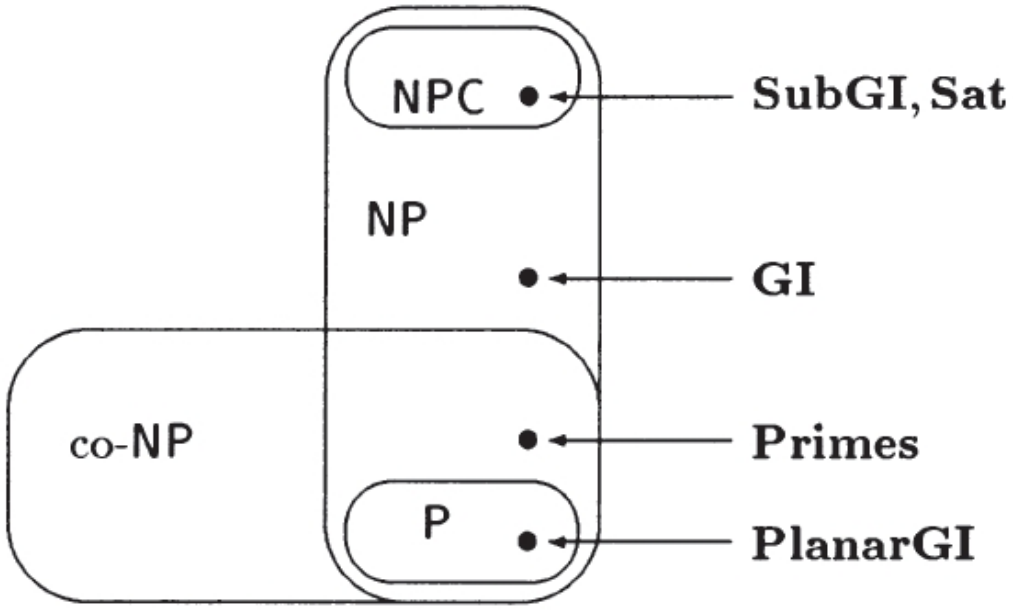
NPC
SubGI, Sat
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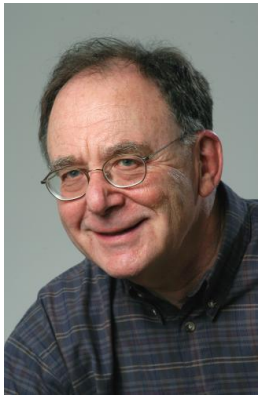
- 

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Comparison of 3 algorithms

Algorithm	Execution Time	Frequency of Failure
Erdös-Babai	$O(n^2)$	$O(n^{-1/7})$
Lipton	$O(n^6 \log n)$	c^{-n}, $c > 1$
Karp	$O(n^2 \log n)$	$O(n^{3/2}2^{-n/2})$

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