Measuring thick monoids

Frenkel Elizaveta
Moscow State University, Università degli Studi di Firenze

joint work with V.N.Remeslennikov
ALMAZ, Omsk, 2015

Introduction

- **Motivation**
  - To explain results of:
    ⊛ E.F, V.N. Remeslennikov, *Special automata over free groups,* in preparation.
- **Previously known**
  - ⊛ A. V. Borovik, A. G. Myasnikov and V. N. Remeslennikov, **Multiplicative measures on free groups**, *Intern. J. of Algebra and Computation*, 13 (2003), **6**, pp. 705 – 731
  - ⊛ E. Frenkel, A. G. Myasnikov and V. N. Remeslennikov, **Regular sets and counting in free groups**, in *Combinatorial and Geometric Group Theory*, Series "Trends in Mathematics", (Birkhauser Verlag Basel/Switzerland, 2010), pp. 93–118
- **Objects and methods**
  - Free groups, regular subsets.
  - Asymptotic and algorithmic methods.

Introduction

- **Motivation**
  - To explain results of:
    ⊛ E.F, V.N. Remeslennikov, *Special automata over free groups,* in preparation.
- **Previously known**
  - ⊛ A. V. Borovik, A. G. Myasnikov and V. N. Remeslennikov, **Multiplicative measures on free groups**, *Intern. J. of Algebra and Computation*, 13 (2003), **6**, pp. 705 – 731
  - ⊛ E. Frenkel, A. G. Myasnikov and V. N. Remeslennikov, **Regular sets and counting in free groups**, in *Combinatorial and Geometric Group Theory*, Series "Trends in Mathematics", (Birkhauser Verlag Basel/Switzerland, 2010), pp. 93–118
- **Objects and methods**
  - Free groups, regular subsets.
  - Asymptotic and algorithmic methods.

Introduction

- **Motivation**
  - To explain results of:
    ⊛ E.F, V.N. Remeslennikov, *Special automata over free groups,* in preparation.
- **Previously known**
  - ⊛ A. V. Borovik, A. G. Myasnikov and V. N. Remeslennikov, **Multiplicative measures on free groups**, *Intern. J. of Algebra and Computation*, 13 (2003), **6**, pp. 705 – 731
  - ⊛ E. Frenkel, A. G. Myasnikov and V. N. Remeslennikov, **Regular sets and counting in free groups**, in *Combinatorial and Geometric Group Theory*, Series "Trends in Mathematics", (Birkhauser Verlag Basel/Switzerland, 2010), pp. 93–118
- **Objects and methods**
  - Free groups, regular subsets.
  - Asymptotic and algorithmic methods.

Part I. Regular subsets of fg free groups

Let $X = \{x_1, \ldots, x_m\}$ be a finite alphabet, M be a monoid over X with identity ϵ. Let $F = F(X)$ be the free group generated by X, i.e. all reduced words over X.

A Nondeterministic Finite Automaton $\mathcal{A}$ over monoid M is a finite directed graph (possibly with multiple edges and loops) with initial and finite states where the arcs are labeled with elements of M.

Reading the labels of paths from initial to final states defines the accepted language $L(\mathcal{A}) \subseteq M$.

$L \subseteq M$ is **regular** if $L = L(\mathcal{A})$ for some NFA over M.

$\lambda-$measure

Let R be an arbitrary subset of the free group F and let S_k be the sphere of radius k in the Cayley graph $C(F, X)$.

Denote by $f_k = \frac{|R \cap S_k|}{|S_k|}$. We shall call f_k relative frequencies of R in F.

We define $\lambda-$measure on subsets of F as

$$\lambda(R) = \sum_{k=0}^{\infty} f_k(R).$$

We say R is $\lambda-$measurable, if $\lambda(R)$ is finite.

A set R is termed exponentially $\lambda-$measurable, if $f_k(R) \leq q^k$ for all sufficiently large k and positive constant $q < 1$.

Generating function of a regular set

The generating function for R is a formal series in $R[[t]]$:

$$g_R(t) = \sum f_k t^k.$$

Theorem *For a regular set $R \subseteq F$ the function $g_R(t)$ is a rational function of t with rational coefficients and either*

- *has no singularity at $t = 1$ (in this case R is exponentially $\lambda-$measurable) or*
- *has a simple pole at $t = 1$ (in this case R is thick).*

In particular,

$$Res_1 g_R(t) = -\mu_0(R). \qquad (1)$$

Algorithm I: Generating function of a regular set

Let A be adjacency matrix of $\mathcal{A}$ such that entries a_{ij} $(i, j = 1, \ldots, n)$ corresponds to the number of arrows from the state i to the state j. Denote by R the subset of F accepted by $\mathcal{A}$.

Algorithm I:

1. Given an automaton $\mathcal{A}$, compute the adjacency matrix A.
2. Compute the fundamental matrix $B = tA(E - tA)^{-1}$ of $\mathcal{A}$, with b_{ij} in the ring of formal power series $R[[t]]$.
3. The generating function $g_R(t)$ is equal to $\sum\limits_{i,j=\overline{1,n}} b_{ij}$.

NB: Involves the matrix inversion and therefore slow for the size of automaton big enough.

Asymptotic classification of sets

A regular subset R of F is called thick, if μ_0 is strictly positive.

Theorem 1.[Borovik, Miasnikov, Remeslennikov; Averina, F.] Let F be a free group and R be a regular subset of F. Then

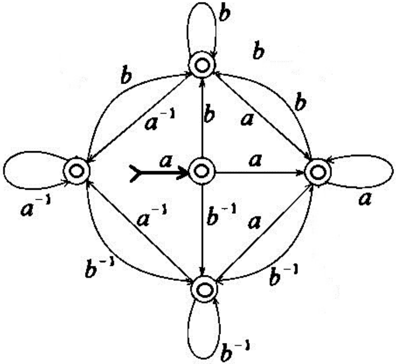
1.) every regular subset in F is either thick or exponentially $\lambda-$measurable;

2.) a regular subset in F is thick if and only if its prefix closure contains a cone.

Cones

A cone $C(w)$ with the handle w is a set of all elements in F containing the given word w as initial segment.

Example. $F = F(a, b)$ and $C(a)$:



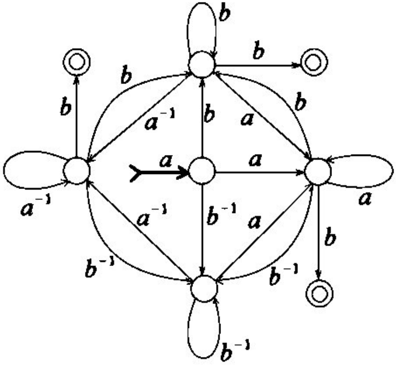
(S)cones

A cone with a right-hand side handle, i.e. the set of all words in F that terminates with w (we denote this sort of cones by $C[w]$).

Cones

A double-based cone with handles w_1, w_2, is a set of all elements in F of the form $w_1 \circ f \circ w_2$, $f \in F$.

Example. $F = F(a, b)$ and $C(a, b)$:



Part II

Regular sets under the microscope:

- **Structure**
- **Thick sets**
- **Decomposition of special automata**

Special automata

Let $\mathcal{A} = (S, X \cup X^{-1}, \delta, i_0, Z)$ be a deterministic automaton. $\mathcal{A}$ is called special over F if

a. The initial vertex has no inedges;

b. There is only one final state $z_0 \in Z$;

c. $\mathcal{A}$ does not contain inaccessible states;

d. For every state $s \in S$ there is a direct path from s to the final state z_0;

e. For any state $s \in S$, all arrows which enter s have the same label $x \in X \cup X^{-1}$. We shall say in this situation that *s has type x*.

f. For any state s of type x in $\mathcal{A}$, all arrows exiting from s cannot have label x^{-1}.

Decomposition into special sets

Theorem 2. *Let* L *be a regular language in* F*. Then there exist a finite number of special automata* $\mathcal{A}_1, \ldots, \mathcal{A}_k$ *such that* L *is a disjoint union of languages* $L_0 = L(\mathcal{A}_0), \ldots, L_k = L(\mathcal{A}_k)$ *in* F*:*

$L = L_0 \sqcup L_2 \sqcup \ldots \sqcup L_k.$

Further splitting

A regular subset $R = R(\mathcal{A})$ of a free group F is tight, if $i_0 = z_0$, and loose otherwise.

Lemma 3. *Let $R = R(\mathcal{A})$ and $\mathcal{A} = (S, X \cup X^{-1}, , i_0, z_0)$ be a special automaton over F. Then there exist regular languages R_1, R_2, R_3 in F such that R_1 is accepted by a special automaton and*

1. *if $\mathcal{A}$ has at least one arrow exiting z_0, then R_2 is non-empty and tight, R_3 is loose and*

$$R = R_1 \circ R_2 \text{ is unambiguos}; \tag{2}$$

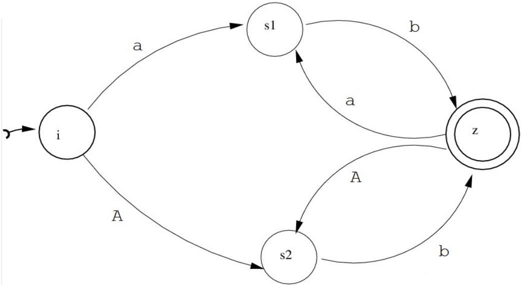
$$R_2 = 1 \sqcup R_3 \sqcup (R_3 \circ R_3) \sqcup (R_3 \circ R_3 \circ R_3) \sqcup \cdots ; \tag{3}$$

$$g_R(t) = g_{R_1}(t) g^*_{R_2}(t); \quad \lambda(R) = \lambda(R_1)\lambda^*(R_2). \tag{4}$$

2. *if there is no arrows exiting z_0, then $R_2 = R_3 = \emptyset$, $R = R_1$, $\lambda(R) = \lambda(R_1)$, and $g_R(t) = g_{R_1}(t)$.*

Further splitting: example

Example. Let $X = \{a, b\}$ be an alphabet and $X \cup X^{-1} = \{a, b, A, B\}$ Consider the special automaton $\mathcal{A}$ (the arrow with a tale corresponds to the initial state, and the finale state is drawn as a double circle).



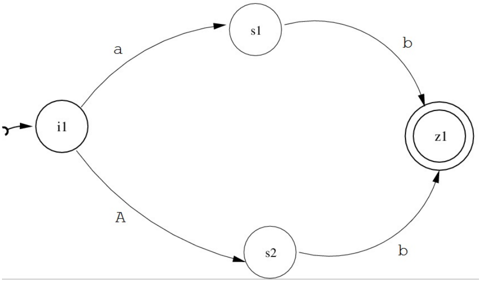
The special automaton $\mathcal{A}$.

$$R = R(\mathcal{A}) = ab\left((Ab)^*(ab)^*\right)^* \cup ay\left((ab)^*(Ab)^*\right)^* \bigsqcup$$

$$\bigsqcup Ab\left((ab)^*(Ab)^*\right)^* \cup Ab\left((Ab)^*(ab)^*\right)^* .$$

Further splitting: example

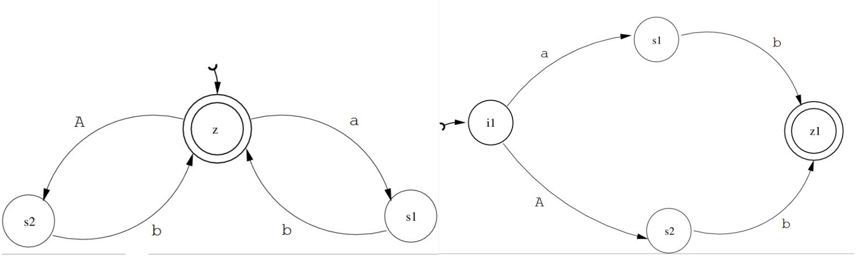
The set of first type $R_1 = ab \cup Ab$.



The special automaton $\mathcal{A}_1$.

Further splitting: example

The sets of second type $R_2 = ((ab^*(Ab)^*)^*$ and third type $R_3 = ab \cup Ab$ with their automata $\mathcal{A}_2$ and $\mathcal{A}_3$.



The automata $\mathcal{A}_2, \mathcal{A}_3$.

Thickness and speciality

It turns out that for regular sets accepted by special automaton their thickness can be easily checked.

An automaton $\mathcal{A}$ accepting a tight set is called *$X-$complete* if for every state $s \in S$ of type x all arrows labeled by $X \cup X^{-1} \setminus \{x^{-1}\}$ exit from s.

Proposition 4. *Let $\mathcal{A}$ be a special automaton, and $R = L(\mathcal{A})$ be a saturated set such that $R = R_1 \circ R_2$ is the splitting of the form above, with $R_1 = L(\mathcal{A}_1)$ and $R_2 = L(\mathcal{A}_2)$. If $\mathcal{A}_2$ is not X-complete, then R is exponentially $\lambda-$ measurable.*

Thick monoids

How does thick monoids look like

This is **not** a thick monoid

Thickness and speciality

Indeed, let R_2 be a regular subset of F of second type accepted by the automaton $\mathcal{A}_2$.

Then R_2 forms a monoid, and if $\mathcal{A}_2$ is $X-$complete, then the **monoid** R_2 is called **thick**.

An interesting fact about thick monoids is that we can describe them in terms of double-based cones. But not the only one!

Lemma 5. *Let R_2 be a thick monoid in F. Then*

1. $R_2 = C[x] \setminus C(x^{-1}, x) \cup \{1\}$ *for some* $x \in X \cup X^{-1}$,
2. $g_{R_2}(t) = \dfrac{(2m-1)t^2}{4m^2(1-t)} + 1 + \dfrac{t}{2m} + \dfrac{t^2}{4m^2} + \dfrac{t^3}{2m(2m-1-t)}$, *and*
3. $\mu_0(R_2) = \dfrac{2m-1}{4m^2}$.

Double-based cones

Lemma 6. *Let $C(a,b)$ be a double-based cone with both handles a,b in $X \cup X^{-1}$. Then following holds:*

1. $f_k(C(a,b)) = f_k(C(c,d))$ *and therefore* $g_{C(a,b)}(t) = g_{C(c,d)}(t)$ *for all* a,b,c,d *in* $X \cup X^{-1}$ *such that* $ab \neq 1$, $cd \neq 1$. *Further,* $f_k(C(a,a^{-1})) = f_k(C(b,b^{-1}))$ *for arbitrary* $a,b \in X \cup X^{-1}$.

2. $f_k(C(a,a^{-1})) = (2m-1)f_k(C(a,a)) - \dfrac{1}{2m(2m-1)^{k-1}}$, *for* $k \geq 3$,

3. $g_{C(a,a)}(t) = \dfrac{t^2}{4m^2(1-t)} + \dfrac{t^2}{4m^2(2m-1)} + \dfrac{t^3}{2m(2m-1)(2m-1-t)}$,
   and $g_{C(a,a^{-1})}(t) = \dfrac{t^2}{4m^2(1-t)} - \dfrac{t^2}{4m^2} - \dfrac{t^3}{2m(2m-1-t)}$,

4. $\mu_0(C(a,b)) = \mu_0(C(c,d)) = \dfrac{1}{4m^2}$ *for all* $a,b,c,d \in X \cup X^{-1}$.

More results

Theorem 7. *Let $R = C(u, v)$ be a double-based cone with handles u, v in F such that $u = u_0 \circ a$, $v = b \circ v_0$, where $u_0, v_0 \in F$ and $a, b \in X \cup X^{-1}$. Then*

1. $g_R(t) = g_{C(a,b)}(t) \cdot \lambda^*(u_0) \cdot \lambda^*(v_0)$;
2. $\mu_0(R) = \dfrac{\lambda^*(u_0) \cdot \lambda^*(v_0)}{4m^2}$.

Theorem 8. *A regular subset R of F is thick if and only if it contains a double-based cone $C(w, x)$, for some $x \in X \cup X^{-1}$ and $w \in F$.*