Приближенное решение задачи управления поставками продукции

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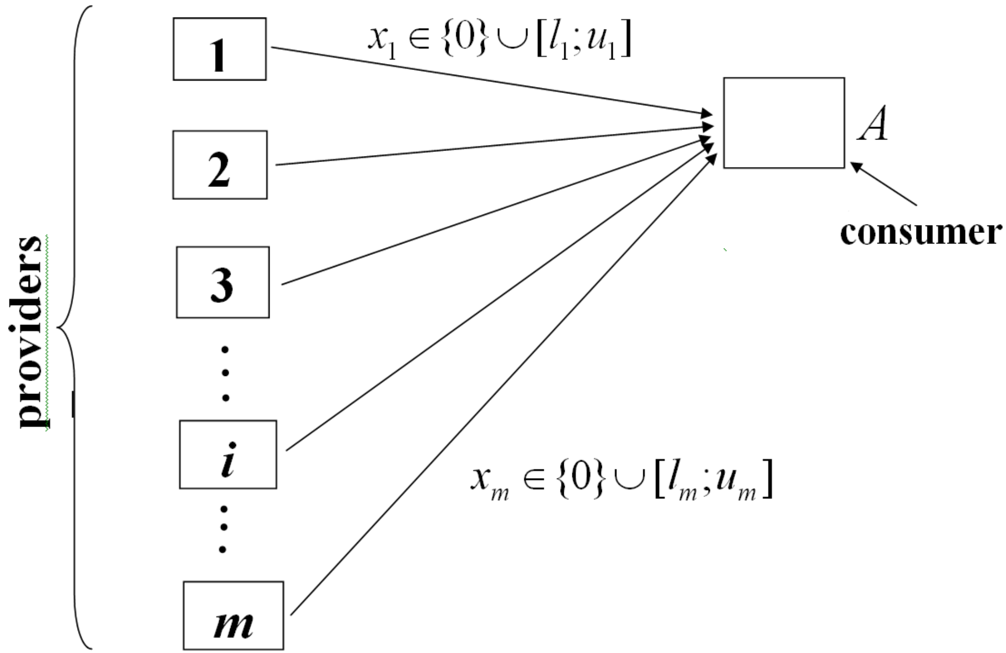
Outline of the talk

- Supply Scheduling Problem (SSP)
  - Formulation of the SSP
  - Hardness of the SSP
- Optimal Solutions
  - A structural property of optimal solutions
  - Dynamic Programming algorithm
- From Dynamic Programming to FPTAS
  - FPTAS for the case of continuous costs
  - FPTAS for the case of concave costs
- Conclusions, questions for further research

The Supply Scheduling Problem (SSP)



SSP as Mathematical Programming Problem

$$\text{Min} \quad f(x) = \sum_{i=1}^{m} c_i(x_i), \tag{1}$$

$$\sum_{i=1}^{m} x_i \geq A, \tag{2}$$

$$x_i \in \{0\} \cup [l_i, u_i], \; i = 1, ..., m. \tag{3}$$

- m is the number of suppliers;
- x_i is quantity of delivery from supplier i;
- $A \geq 0$ is the demand of the consumer;
- $l_i \geq 0$ is the *minimum* quantity of delivery from supplier i;
- $u_i \geq m_i$ is the *maximum* quantity the supplier i;
- $c_i(\cdot)$ is the non-decreasing efficiently computable function representing the shipment cost from supplier i, $i = 1, \ldots, m.$

W.l.o.g, we assume that $\sum_{i=1}^{m} u_i \geq A,$ so that SSP has a solution.

Survey of Results from

1. Еремеев А.В., Ковалев М.Я., Кузнецов П.М. Приближенное решение задачи управления поставками со многими интервалами и вогнутыми функциями стоимости // *Автоматика и телемеханика* – 2008. – № 7. – С. 90–97.

2. Еремеев А.В., Романова А.А., Сервах В.В., Чаухан С.С. Приближенное решение одной задачи управления поставками // *Дискретный анализ и исследование операций.* Сер. 2. – 2006. – Т. 13. – № 1. – С. 27–39.

3. Chauhan S.S., Eremeev A.V., Romanova A.A., Servakh V.V., Woeginger G. J. Approximation of the supply scheduling problem // *Operations Research Letters* – 2005. – Vol. 33. – N 3. – P. 249–254.

4. Ng C.T., Kovalyov M.Y., Cheng T.C.E. An FPTAS for a supply scheduling problem with non-monotone cost functions // *Naval Research Logistics.* – 2008. – Vol. 55. – P. 194–199.

Hardness of the SSP

Proposition 1. (Chauhan, Eremeev, Romanova, Servakh, 2004)
The SSP *is* NP*-hard.*

Proof. The NP-hard MINIMUM KNAPSACK problem

$$\text{Min} \sum_{i=1}^{n} a_i z_i, \tag{4}$$

$$\sum_{i=1}^{n} b_i z_i \geq B, \tag{5}$$

$$z_i \in \{0, 1\},\ i = 1, \dots, n \tag{6}$$

reduces to SSP assuming $m = n,\ A = B$ and

$$c_i(x) = a_i x_i,\ \ l_i = u_i = b_i,\ \ i = 1, \dots, n.$$

A structural property of optimal solutions

Proposition 2. (Ng, Kovalyov, Cheng, 2008) *If* $c_i(x_i)$ *are concave and non-decreasing on* $[l_i, u_i], i = 1, \ldots, m$, *then* SSP *has an optimal solution where* $x_i \in \{0, l_i, u_i\}$ *for all suppliers* $i = 1, ..., n$, *except for at most one supplier* j, *for which* $l_j < x_j < u_j$ *holds.*

This property may be extended to the case where each supplier has a number of feasible intervals for shipment size (Eremeev, Kovalyov, Kuznetsov, 2008).

Complete Enumeration Algorithm

In conditions of Proposition 2, the optimal solution can be found in $O(m3^{m-1})$ time using the following algorithm (Proth, 2004):

1. Set $f_{\text{best}} = \infty$.
2. For each "free" variable index $j = 1, \ldots, m$ do:
2.1. Loop over all 3^{m-1} combinations of $x_i,\ i \neq j$, such that
$x_i \in \{0, l_i, u_i\}\ \forall\ i \neq j$, performing Steps 2.1.1, 2.1.2:
2.1.1. Take $x_j = A - \sum_{i \neq j} x_i$
2.1.2. If $r = c_j(x_j) + \sum_{i \neq j} c_i(x_i) < f_{\text{best}}$, then
update the best-found solution: $f_{\text{best}} = r,\ x_{\text{best}} = x$
3. If $f_{\text{best}} < \infty$ then x_{best} is the optimum.

Supplementary Optimization Problem

The enumeration in Step 2.1 can be substituted by solving a supplementary optimization problem.

Given the index of a "free" variable $\boldsymbol{j}$, at the main loop, we need to solve:

$$\text{Min} \sum_{1 \leq i \leq m,\, i \neq j} \boldsymbol{c_i(x_i)}, \tag{7}$$

$$\sum_{1 \leq i \leq m,\, i \neq j} \boldsymbol{x_i = A}, \tag{8}$$

$$\boldsymbol{x_i \in \{0, l_i, u_i\}, \;\; 1 \leq i \leq m, \; i \neq j.} \tag{9}$$

Dynamic Programming (Chauhan, Eremeev, Romanova, Servakh, 2004)

Consider the subproblem for $j = m$ and $c_i(\cdot) \in \mathbf{Z}\ \forall i$:

$$\text{Min} \sum_{1 \le i \le m-1} c_i(x_i), \tag{10}$$

$$\sum_{1 \le i \le m-1} x_i = A, \tag{11}$$

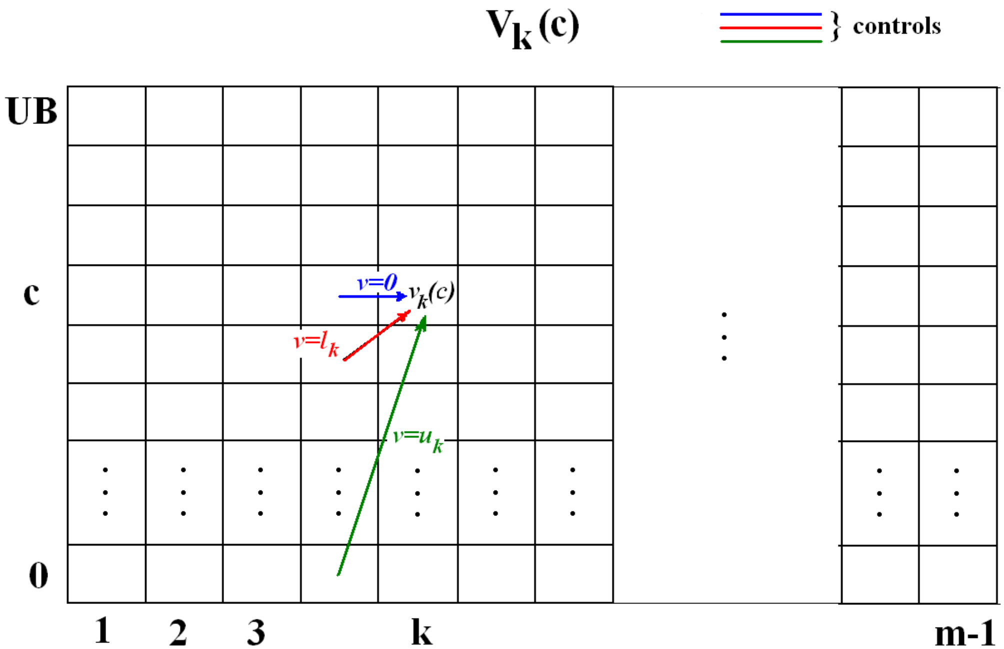
$$x_i \in \{0, l_i, u_i\},\ \ 1 \le i \le m-1. \tag{12}$$

Let $V_k(c)$ denote the maximal *amount* the first k suppliers can provide at cost not greater than c, so that (12) is satisfied.

Subproblem (10)–(12) is solvable in $O(mUB)$ time by Bellman equation:

$$V_k(c) = \max\left\{v + V_{k-1}(c - c_k(v))\ :\ v \in \{0, l_k, u_k\},\ c_k(v) \le c\right\},$$

$$k = 1, \ldots, m-1,\ c = 0, \ldots, UB,\ \text{where } UB := \sum_{i=1}^{m} c(u_i).$$



$$V_k(c) = \max\left\{v + V_{k-1}(c - c_k(v)) \;:\; v \in \{0, l_k, u_k\},\; c_k(v) \le c\right\},$$

$$k = 1, \ldots, m-1,\; c = 0, \ldots, UB, \text{ where } UB := \sum_{i=1}^{m} c(u_i).$$

Standard Definitions for Approximation Algorithms

ρ-*approximation algorithm* is an algorithm that returns a feasible solution with cost at most ρ times exceeding the minimum cost (if the problem is solvable).

By a *fully polynomial time approximation scheme* (FPTAS) we mean a family of polinomial-time $(1+\varepsilon)$-approximation algorithms over all $\varepsilon > 0$, running in polinomial time of $1/\varepsilon$.

DP-Based FPTAS: Rounding-Input-Data

Rounding-the-input-data (Sahni, 1976; Gens, Levner, 1979).

In our case the costs $c_i(x_i)$ are substituted by

$$c_i'(x_i) = \lfloor c_i(x_i)/\delta \rfloor,\ i = 1, \ldots, m, \tag{13}$$

where δ is a scaling factor:

- δ is sufficiently *large* to make $c_i'(\cdot)$ polynomially bounded;
- δ is sufficiently *small* to ensure $(1 + \varepsilon)$-approximation.

Rounding-the-Input for SSP

Let $L \le f^*(x) \le U$ for all feasible x, e.g.

$$L := \min\{c_i(l_i), i = 1, \ldots, m\},$$

$$U := \sum_{i=1}^{m} c(u_i).$$

Choosing $\delta := \varepsilon L/(m-1)$ and rounding the input for the above DP gives

Theorem (Ng, Kovalyov, Cheng, 2008). *If every function* $c_i(\cdot)$ *is concave on* $[l_i; u_i]$ *then there exists an FPTAS for* SSP *with* $O(m^3 \log_2 \log_2(U/L) + m^3/\varepsilon))$ *time complexity.*

This result may be generalized to the case when several admissible intervals are given for each $i = 1, \ldots, m$ (Eremeev, Kovalyov, Kuznetsov, 2007).

DP-Based FPTAS: Trimming-State-Space

Trimming-the-state-space (Ibarra, Kim, 1976): group together the 'close' states of DP in order to reduce the size of state space down to polynomial.

Woeginger G. J. When does a dynamic programming formulation guarantee the existence of a fully polynomial time approximation scheme (FPTAS)? // *INFORMS Journal of Computing.* – 2000. – Vol. 12. – N 1. – P. 57–74.

Trimming the State Space for SSP

Group the DP-states, i.e. partial solutions $\boldsymbol{x}_{part} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$, with 'close' costs $\boldsymbol{C}_k$ and equal volumes $\boldsymbol{V}_k$.

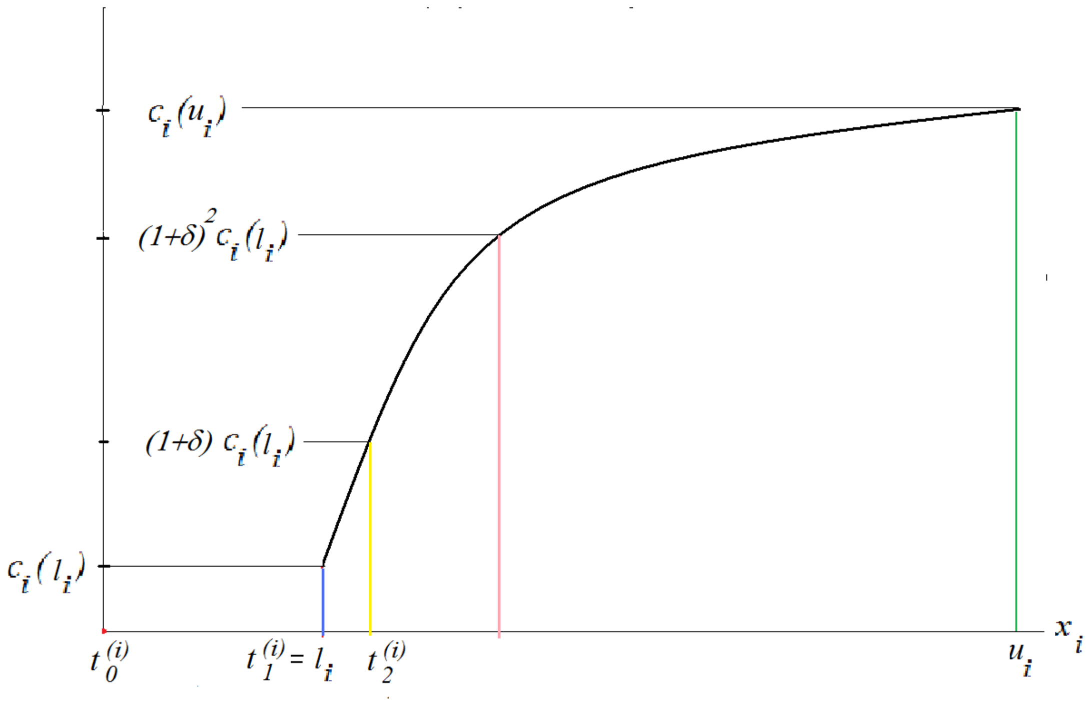
Trimming the state space with parameter $\boldsymbol{\Delta = 1 + \epsilon/6m}$.:

A state $\boldsymbol{x}_{part} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$ is $\boldsymbol{\Delta}$−dominated by another state $\boldsymbol{x}'_{part} = (\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_k)$, iff

$$\boldsymbol{V}_k \leq \boldsymbol{V}'_k \qquad \text{and} \qquad \boldsymbol{\Delta} \cdot \boldsymbol{C}_k \geq \boldsymbol{C}_k. \tag{14}$$

That is, the dominating state $\boldsymbol{x}'_{part}$ delivers a quantity that is at least as large as the quantity in the dominated state $\boldsymbol{x}_{part}$, whereas the cost of $\boldsymbol{x}'_{part}$ is (at most) slightly higher than the cost of $\boldsymbol{x}_{part}$.

Polynomial Number of Controls for SSP



Trimming the State Space for SSP

Theorem (Chauhan, Eremeev, Romanova, Servakh, Woeginger, 2005). *If for all* $i = 1, \ldots, m$

- $c_i(l_i) = \Omega(1)$,
- $c_i(\cdot)$ *is continuous on* $[l_i, u_i]$,
- $\max\{y \in [l_i, u_i] \; : \; c_i(y) \leq t\}$ *is computable in time* $O(1)$ *for* $t > 0$,

then there is an FPTAS for the SSP *with time complexity*

$$O(m^2 \log_2 m/\varepsilon^2 \cdot \log_2^2(U/L).$$

The result holds for more general case including continuous *non-monotone* cost functions and forbidden areas in $[l_i, u_i]$.

A slightly faster FPTAS based on Rounding-the-Input is developed in (Ng, Kovalyov, Cheng, 2008).

Summary on Approximation algorithms

Concave non-decreasing costs:

	FPTAS I[a]	FPTAS II[b]
intervals	single	multiple (up to I_{max})
time	$O\left(m^3 \log_2 \log_2(\frac{U}{L}) + \frac{m^3}{\varepsilon}\right)$	$O\left(I_{max} m^3 (\log_2(\frac{U}{L}) + \frac{1}{\varepsilon})\right)$

Continuous costs:

FPTAS III[c]	FPTAS IV[d]
$O\left(\frac{m^2 \log_2 m}{\varepsilon^2} \log_2^2 \frac{U}{L}\right)$	$O\left(\frac{m^2}{\varepsilon^2} \log_2\left(\frac{U}{L}\right) + \log_2 \log_2 \frac{U}{L}\right)$

[a] Ng C.T., Kovalyov M.Y., Cheng T.C.E., 2008.
[b] Eremeev A.V., Kovalyov M.Y., Kuznetsov P.M., 2007.
[c] Chauhan S.S., Eremeev A.V., Romanova A.A., Servakh V.V., Woeginger G.J., 2005.
[d] Ng C.T., Kovalyov M.Y., Cheng T.C.E., 2008.

Conclusions

- SSP is NP-hard but solvable by pseudo-polynomial time DP.
- The methods of Rounding-the-Input and Trimming-the-State-Space are both applicable to transform the DP into FPTAS for SSP.
- In the case of concave costs the time complexity bound of FPTAS is less dependent on ε than in the general case of continuous costs.

Questions for further research

- Is it possible to extend the results to the multiple planning periods case?
- Is it possible to extend the results to the multiple products case?
- What if the delivery cost functions have a bounded number of discontinuity points?
- Is it possible to improve the known results on MINIMUM NONLINEAR KNAPSACK problem (Halman, Klabjan, Li, Orlin, Simchi-Levi, 2014) using the FPTASes III and IV?

Thank you for your attention!